CONTINUITY IN SYNTHETIC GEOMETRY.

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1. The geometry of Euclid was a geometry of particular cases. Its main characteristics were the logic of its argument and the sequence of its propositions. The relationship between one theorem and another was only that of logical dependence; and there was no attempt to find general principles which might underlie the separate theorems. The same was true of the later geometry of the conic sections.

A certain amount of generalization has now been attained through the principle of continuity as suggested first by Kepler and developed more fully in the projective geometry of Poncelet. This principle involved two ideas,—1st. the idea of infinity which led to that conception of the conic sections in which the parabola occupies a boundary position between the ellipse and the hyperbola, and which made possible more general statements of projective properties; and 2nd. the idea of imaginary points which led to the generalization of theorems to include the cases in which some of the elements involved are imaginary.

The principle of continuity as thus developed has resulted in much that is distinctive in modern geometry. But very little has been done to examine the extent to which the application to elementary geometry of the wider notion of continuity, as applied to mathematical functions in general, will result in the discovery of greater generalization. Work along this line will be found valuable in the teaching of synthetic geometry.

2. The following definitions will serve as a basis:

(i) A plane curve C is the locus of all the possible positions which a variable point P may take, in a definite order, according to some given law.

(ii) Let A and B be two fixed points in C. Then P is said to *move* from A to B when it takes all its positions, in order, from A to B. In such a case P is said to *generate* the segment AB of C.

It will be seen that the movement, or motion, of P, as thus defined, has not necessarily all the characteristics of physical motion.

(iii) Let L be a fixed point in the plane. If the distance PL becomes less than any assigned value, however small, as P moves in C from whatever direction, then L is the *limit*, or the *limiting point*, of the positions of P as it approaches L. (Cf. Pierpont, *Real Variables*, vol. I, § 254.)

Various cases of this may be illustrated as follows: Let AB be a line segment 4 units long, D its middle point, E any other point on it, and P a variable point on it.

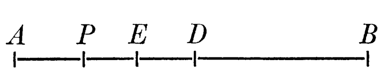

FIG. 1.

(a) Let C be the segment AB. Then any point E on it is the limit of the positions of P as P approaches it. In particular, A and B are limiting points.

(b) Let C be AB with the middle point D not included. That is, as P moves in C it takes every position on AB except D, and AP takes every value from 0 to 4 except the value 2. Then D is the limit of the positions of P as P approaches it. This is an example of a point which is not on C and yet is the limit of points on C.

(c) Let C be AB with the points between E and D not included. That is, as P moves in C it takes every position on AB except those between E and D; and AP takes every value from 0 to AE and from 2 to 4 inclusive. Then E is the limit of the positions of P as it is approached from the left but not as it is approached from the right; and D is the limit as approached from the right but not from the left. Hence at E or D a unique limit, according to the definition, does not exist. If, however, C were AE only, E would be a unique limit.

(iv) If the limit L of the positions of P is itself a position of P, that is if L is on C, then the curve C is *continuous* at L. But at any point where L is not on C, or where no unique limit exists, the curve C is *discontinuous*.

Thus in (iii) (a) above, C is continuous at every point. In (iii) (b) C is discontinuous at D; and in (iii) (c) AE and DB are continuous at every point, but C is discontinuous at E and D.

(v) A curve is continuous over any segment of it when it is continuous at every point of that segment; and the motion of P as it generates the segment is a continuous motion.

3. Motion of a line in the plane.

Let P and P' be two distinct variable points which generate two curves C and C'. Then the line l, which is determined by P and P', is a variable line, and *moves* in the plane. A limit of l is defined as the line LL', when P and P' simultaneously approach distinct limits L and L' on C and C'.

If C'' is any other curve in the plane which intersects l in P'' and LL' in L'', then P'' approaches the limit L'' when P and P' approach the limits L and L'.

The motion of a line in a plane is continuous wherever it actually occupies its limiting position.

The following particular cases may occur:

(i) P and P' move continuously, and l occupies its limiting position LL' at every stage. The movement of l is then continuous.

(ii) P' is a fixed point and P moves continuously on a curve C which does not contain P'; then, in general, l rotates continuously about P'.

(iii) Either P or P' has a point of discontinuity in its motion; l will have, in general, a corresponding position of discontinuity.

(iv) If L and L' become coincident the line LL' is indeterminate in direction, and does not define the limit of l. In this case the limit is defined by LL'' where L'' is the limit of P'' on C''. But as l at this point is indeterminate in direction,

it need not occupy its limiting position LL'', and its motion is therefore discontinuous. The following examples will illustrate this:

(*a*) Let P and P' move on two straight lines c and c' (Fig. 2), which intersect at L, in such a way that the ratio $LP : LP'$ is constant; and let P'' move on a line c'' which does not pass through L. The line l moves parallel to itself; and as P and P' approach coincidence at L, l approaches the limit LL''. But as l is not determined at L, its motion is not continuous there.

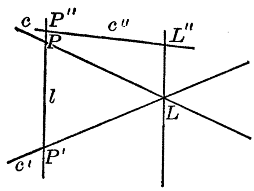

FIG. 2.

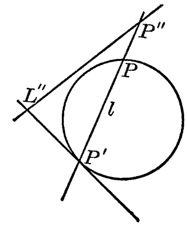

FIG. 3.

(*b*) P moves on a circle c, and P' is a fixed point on c (Fig. 3). As P moves into coincidence with P', l approaches the limiting position $P'L''$ which is defined to be the tangent to the circle at P'. But l is not determined when P is at P', and the rotation of l about P' at this point is therefore discontinuous.

If we fix, by definition, the position of l when P coincides with P' to be that of the tangent at P', we make the rotation of l about P' to be continuous at every point as P moves over the whole circumference.

4. Variable quantities.

The movement of points and lines in the plane implies the variation of quantities such as length, area, and angle. Thus if A of the triangle ABC moves in the plane while B and C remain fixed, the lines AB and AC rotate about B and C; and the lengths AB and AC, the area ABC, and the angles A, B, and C are variable quantities.

The definitions of § 2 practically involve the following definitions concerning variable quantities:

(i) If the value of a variable quantity, as it approaches a fixed value v, may be made to differ from v by less than any assignable quantity however small, whether it approaches through values greater or less than v, then v is the *limit* of the values of the quantity as it approaches v. (Cf. Goursat-Hedrick, *Mathematical Analysis*, vol. I, § 1.)

(ii) If the limit of a variable quantity and its value at a point are the same, the variation is continuous at that point. (Cf. Pierpont, *l. c.*, vol. I, § 339.)

(iii) The variation of a quantity is continuous over any interval when it is continuous at every point of that interval. (Cf. Goursat-Hedrick, *l. c.*, vol. I, § 3.)

Thus for example in Fig. 1 the length AP is continuous over the interval from 0 to AB in 2 (iii) (*a*); but it is discontinuous at D in (*b*), and at E and D in (*c*), as P moves from A to B.

5. In general, continuous movements of points and lines in the plane imply continuity in the variation of the quantities involved. Thus, for example, the distance AP from a fixed point A to a point P which moves continuously along a

curve is a continuous variable. Similarly the angle AOP between a line OP, which rotates continuously about a fixed point O, and a fixed initial position OA is a continuous variable. In each case the variable attains its limiting value when the point or the line reaches its limiting position.

But examples are given in 3 (iv) (*a*) and 3 (iv) (*b*) of continuous movements of points which imply discontinuous movements of lines and hence discontinuous variation in angle. In these cases the points reach their limiting positions where at the same time the angle need not attain its limiting value.

Similarly a continuous rotation of a line may determine a discontinuous movement of a point. Thus let O be a fixed point and l a fixed line (Fig. 4); and let a line m rotate continuously about O and intersect l in P. As m rotates about O, P moves continuously along l except when l and m are parallel. The movement of P and the distance AP are discontinuous at this point.

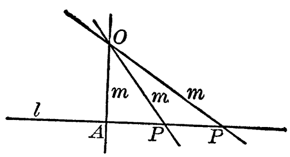

FIG. 4.

On the other hand, if the point at infinity be defined as usual as a unique point on l, the movement of P along l, which is discontinuous when P is the point at infinity, implies a continuous rotation of m about O.

6. In a plane geometric figure some points may be considered fixed while others vary, as for example in the triangle ABC of § 4. Similarly in the quadrangle $ABCD$ we may assume the points A, B, C to remain fixed and D to move freely in the plane. The form of the quadrangle then changes. The angles at C, D, A and the lengths CD and DA are variable quantities, and are functions of D in the sense that their values depend on the position of D. If D moves continuously these quantities vary continuously, except possibly where D coincides with one of the other points. When D approaches a limit, all the variable elements which depend on it approach at the same time their respective limits; and if no discontinuities exist they all have the values of their limits when D occupies its limit.

As D moves along the curve DD'' in Fig. 5, the angle BCD changes to BCD''. In its variation it passes through a zero value when D is at D'; and on the principle that a continuously increasing or decreasing variable changes sign on passing through a zero value, we say that BCD and BCD'' have opposite signs.

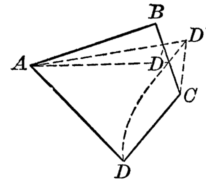

FIG. 5.

7. Suppose a finite set of variables to approach their respective limits at the same time; and suppose a constant relation to exist among these variables for all simultaneous sets of values preceding the limit. Then this relation will exist also at the limit, provided no discontinuities appear. This is evident, for it is merely stated that a certain relation exists among a set of values when it exists for values which differ from them by as little as we please. The principle thus stated is important in that it leads to generalizations among theorems of geometry. The following examples are given in illustration:

(i) ABC is a triangle inscribed in a circle, with A and B fixed, and C variable, on the circumference (Fig. 6). Then, from elementary geometry, the angle ACB is constant for all positions of C provided C remains on either the one side or on the other of AB.

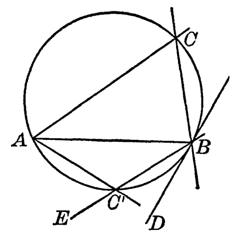

Fig. 6.

But assuming, as in 3 (iv) (*b*), that when C is at B, CB is the tangent BD at B, and that therefore all the elements involved are continuous as C moved into coincidence with B, we have from the above principle that the angle ABD is equal to the constant angle C. Similarly the angle $AC'E$ is seen to be equal to ABD. It follows therefore that the angle ACB is constant as C moves about the whole circumference.

This general statement includes the several theorems usually given in this connection in elementary geometry. It depends however on the assumption that CB occupies its limiting position BD when C comes to B.

The theorems thus connected are:

(*a*) Two angles at the circumference of a circle, standing on the same arc, are equal.

(*b*) The angles between a tangent and a chord from the point of contact are equal to the angles in the alternate segments.

(*c*) An exterior angle of a quadrilateral inscribed in a circle is equal to the opposite interior angle.

(ii) The rectangles on the segments of two intersecting chords of a circle are equal; that is, $PA \cdot PB = PC \cdot PD$.

By moving P continuously, we find that the theorem takes the following additional forms:

(*a*) The square on the ordinate from a point on the diameter of a circle is equal to the rectangle on the segments of the diameter.

(*b*) The rectangles on the segments of two intersecting secants are equal.

(*c*) If a secant and a tangent be drawn from the same point, the rectangle on the segments of the secant is equal to the square on the tangent.

(*d*) The squares on the two tangents from a point are equal.

(iii) The square on a side of a triangle, opposite an acute angle, is equal to the sum of the squares on the other two sides diminished by twice the rectangle on one of these and the projection of the other on it; that is, $AB^2 = BC^2 + CA^2 - 2BC \cdot DC$.

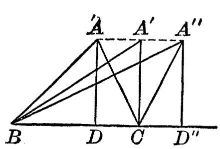

Fig. 7.

The principle of this theorem includes also each of the following:

(*a*) When A moves to D (Fig. 7), the line segment BA is divided externally at C; and for this divided line segment, $BA^2 = BC^2 + CA^2 - 2BC \cdot CA$.

(*b*) When A moves to A', and DC vanishes, the theorem is: The square on the hypotenuse of a right angled triangle is equal to the sum of the squares on the other two sides.

(c) When A moves to A'', $D''C$ is negative and the angle BCA'' is obtuse. Then $A''B^2 = BC^2 + CA''^2 + 2BC \cdot CD''$.

(d) When A moves to D'', the line segment BA is divided internally at C; and for this divided segment $BA^2 = BC^2 + CA^2 + 2BC \cdot CA$.

(iv) $ABCD$ is a complete quadrangle with opposite pairs of sides AB and CD, AC and BD, and AD and BC; and E and F are the middle points of one of these pairs, say BD and AC respectively. Then from elementary geometry,

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4EF^2.$$

This theorem, in principle, includes the following:

(a) When D moves to B, $AB^2 + BC^2 = 2BF^2 + 2AF^2$. That is, the sum of the squares on two sides of a triangle is equal to twice the square on half the third side and twice the square on the median to the third side.

(b) When B and D move to H, any point on the llne-segment AC, $AH^2 + HC^2 = 2AF^2 + 2FH^2$. That is, if a line segment be divided equally at one point and unequally at another, the sum of the squares on the unequal segments is equal to twice the sum of the squares on half the line and on the part between the points of section.

(c) When D moves to C, and E therefore to the middle point G of BC, $AB^2 = 4FG^2$. That is, the square on the base of a triangle is equal to four times the square on the line joining the middle points of the sides.

Various other properties of the triangle and of the divided line segment may be obtained as particular cases of this general theorem.

In the same way also properties of any polygon may be found in special forms as properties of any polygon of a smaller number of sides, or of the divided line segment. Examples (iii) and (vii) further illustrate the same idea.

(v) The three radical axes of three circles taken two and two are concurrent. This theorem takes particular forms as one, two, or three of the circles become points.

If the three circles become points, the theorem states that the right bisectors of the sides of a triangle are concurrent.

(vi) S is a circle, tangent to four mutually external circles S_1, S_2, S_3, and S_4; and t_{12} is a common tangent to S_1 and S_2, etc. Then by Casey's theorem[1]

$$t_{12}t_{34} \pm t_{13}t_{42} \pm t_{14}t_{23} = 0.^2$$

This theorem takes particular forms as one, two, three, or four of the circles become points. Thus, for example,

(a) If the four circles which touch S become points, the result is Ptolemy's theorem on the concyclic quadrangle.

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[1] *Proceedings of the Royal Irish Academy*, 1866. See also J. L. Coolidge, *A Treatise on the Circle and Sphere*, Oxford, 1916, p. 38.—EDITOR.

[2] "Here all the t_{ij}'s denote common direct tangential segments, or those connecting two pairs [of circles] with no common member denote direct tangents and the other four transverse, or those which lack one subscript denote direct, and those which include it transverse tangential segments" (Coolidge).

(*b*) If the four circles which touch S become points and S becomes a straight line, we have the common theorem on four collinear points A, B, C, D, viz: $AB \cdot CD + BC \cdot AD + CA \cdot BD = 0$.

(vii) A, B, C, D, E, F, are six points in any order on a conic section; and $ABCDEFA$ is an inscribed hexagram. If AB and DE, BC and EF, CD and FA intersect in X, Y, Z, respectively, then XYZ is a straight line, the Pascal line of the hexagram.

By moving vertices into coincidence, and assuming that their joins become tangents to the conic, we obtain this theorem in forms adapted to the pentagram, the tetragram, and the triangle.

Thus, for example, the hexagram becomes a triangle if B moves to A, D to C, and F to E. The theorem then becomes: If a triangle be inscribed in a conic, the sides intersect the tangents at the opposite vertices collinearly. The line of collinearity is the Pascal line of the triangle. Or the theorem may be stated thus: If a triangle be inscribed in a conic, and another circumscribed at the vertices of the first, the two triangles are in perspective. The axis of perspective is the Pascal line of the inscribed triangle.

This list of examples may be increased indefinitely from both elementary and advanced geometry. It will be found everywhere that general principles may be discovered to underlie groups of theorems which in our ordinary teaching of the subject have had no relationship to one another.

A NEW PROOF OF THE LAW OF TANGENTS.[1]

By WM. F. CHENEY, Jr., Berkeley, Cal.

1. In the triangle ABC, with angles A, B, and C, and sides a, b, and c, assume b greater than c. Lay off AD along b, equal to c, and draw BD; then

$$\angle DBA = \tfrac{1}{2}(180^\circ - A) = \tfrac{1}{2}(B + C);$$

[1] Other geometrical discussions of the law of tangents may be found in the following sources: W. E. Johnson, *Treatise on Trigonometry*, London, 1889, p. 96; R. Levett and C. Davison, *Elements of Plane Trigonometry*, London, 1892, pp. 170–171 (also in J. W. Mercer, *Trigonometry for Beginners*, Cambridge, 1906, pp. 259–260); E. Brand, *Journal de mathématiques élémentaires* (de Longchamps), 1895, pp. 153–154; E. M. Langley, *Journal de mathématiques élémentaires* (de Longchamps), 1896, pp. 3–4 (construction introducing Wallace's Line); E. W. Hobson, *A Treatise on Plane Trigonometry*, second edition, Cambridge, 1897, pp. 155–156; E. J. Wilczynski, *Plane Trigonometry and Applications*, Boston, 1914, pp. 105–106; and J. W. Young and F. M. Morgan, *Plane Trigonometry and Numerical Computation*, New York, 1919, pp. 47–48.

The law of tangents in the usual form (except for notation) was first given by Vieta in the seventeenth century; *Francisci Vietæ opera mathematica*, Lugduni Batavorum, 1646, p. 402: "Vt adgregatum crurum ad differentiam eorundem, ita prosinus dimidiæ summæ angulorum ad basin ad prosinum dimidiæ differentiæ."

Cf. A. von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie*, Teil 1, 1900, p. 188; Teil 2, pp. 44–45; and other places referred to under heading "Tangentensatz" in indexes.—Editor.